Determining the Distance and Size of the Moon from Eclipse Observations

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1. Introduction.

In this activity we will use measurements of the times at which certain phenomena occur during a lunar eclipse to determine the distance between the Earth and the Moon, and the size of the Moon. We will use an approach developed by ancient Greek astronomers more than 2000 years ago. This activity assumes that you have read through the first eclipse lab, "Observing an Eclipse of the Moon", and are familiar with the concept of a lunar eclipse. This lab can be done together with the first one. The only additional work during the eclipse itself is to carefully observe the umbral shadow and note the precise times when it crosses the middle of the Moon, near the beginning and end of umbral eclipse.

We will use a method first invented by Aristarcos (c. 270 BCE) and developed further by the greatest astronomer of Hellenic Greece, Hipparcos (c. 140 BCE). They didn't have telescopes or computers, just their unaided eyes, the ability to measure time, and a knowledge of geometry. We will use the same tools. We will use two different pictures, or **models,** of the circumstances of an eclipse. The first part of the lab uses shadow timings made during the eclipse, an approximate model, and a little geometry to get an approximate result for the distance to the Moon. In the second part of the lab, we will make a measuring device out of simple materials and use it to measure the angular size of the Sun. We'll combine this with the shadow timings and somewhat more complicated geometry to get a more accurate result for the Moon's distance, and an estimate of its size.

2. First method: the cylindrical shadow model

Recall from the first lab activity "Observing an Eclipse of the Moon" how a lunar eclipse occurs when the Moon passes through the Earth's cone-shaped shadow:

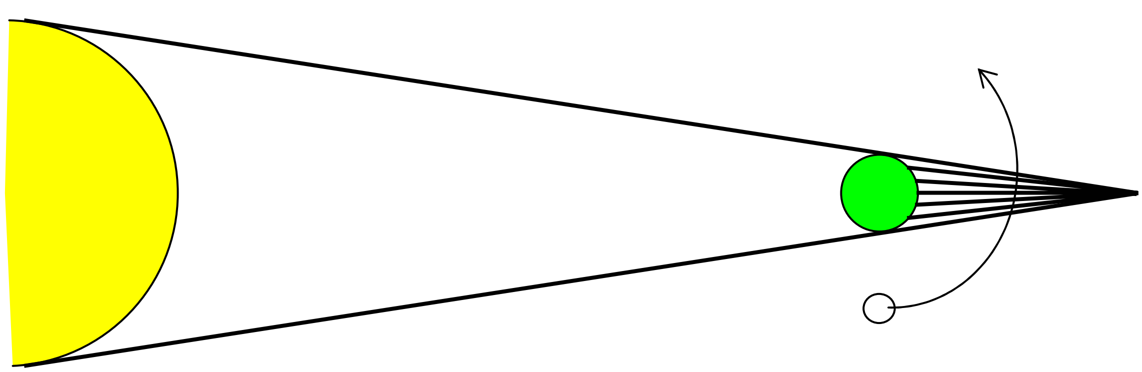


Fig. 1. Basic geometry of a lunar eclipse.

Now we know (and the ancient Greeks knew too) that the Sun is much further from the Earth than indicated in this simple diagram. If we move the Sun off to the left, the conical shadow of the Earth gets longer, and the sides of the shadow in our figure become more nearly parallel:

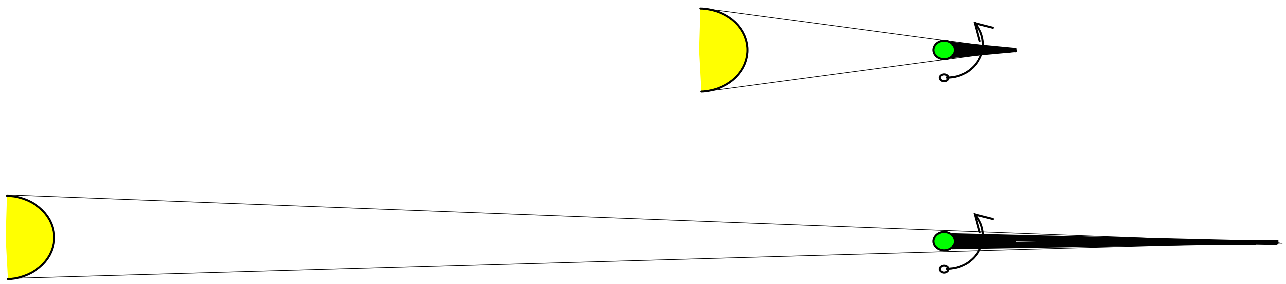
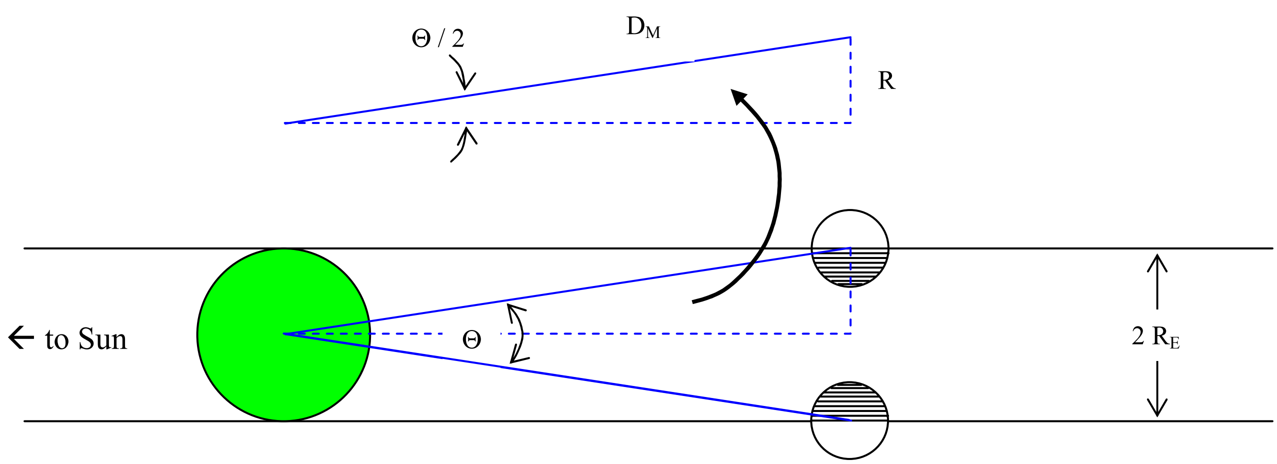


Fig. 2. Effect on shadow geometry of placing the Sun at a greater distance.

For the cylindrical shadow model, we assume that the Sun is infinitely far away. Then the Earth's shadow becomes a *cylinder*, the same size as the Earth, extending indefinitely out into space. This is an approximate model with very simple geometry. If we know the angle Θ that the shadow makes where the Moon crosses it, we can determine the distance to the Moon D_M in units of the Earth's radius R_E :

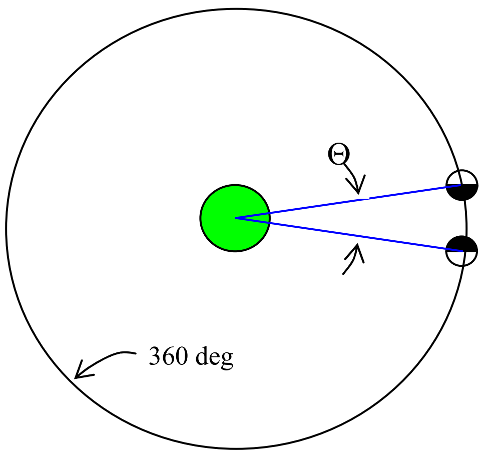


$$R_E / D_M = \sin [\, \Theta / 2 \,] \qquad \text{so} \qquad D_M = R_E / \sin [\, \Theta / 2 \,]$$

Fig. 3. Geometry and formula for determining the distance of the Moon in the cylindrical shadow model.

How can we measure the angle Θ ? We can't see the shadow itself extending out in space, and we can't see the shadow on the Moon at the two positions shown in Figure 3 at the same time.

What we *can* do is measure how long it takes the Moon to pass through the Earth's shadow. The Moon goes completely around the Earth, making a 360 degree angle, in one lunar month of 27.322 days[1]. So the time it takes to pass through the shadow $\mathbf{T_S}$, compared to 27.322 days, is the same as the shadow's angular size compared to 360 degrees:



$$\frac{\mathbf{T_S}}{\mathbf{27.\,322\ days}} = \frac{\mathbf{\Theta}}{\mathbf{360\ deg}}$$

$$\mathbf{\Theta\ \ (deg) = 360\ x\ [\ T_S\ /\ 27.\,322\ days\ ]}$$

Fig. 4. Relation between shadow angular size Θ and time $\mathbf{T_S}$ for the Moon to pass through it.

For this part of the activity, we will measure the length of time $\mathbf{T_S}$ it takes the Moon to cross the Earth's shadow, use this to determine the angle Θ, and then use this to get the distance of the Moon in units of the Earth's radius. We can see from Figures 3 and 4 that we need to write down the time when the umbral shadow is just crossing the center of the Moon, as the Moon enters and then leaves the umbra. This requires careful observation as the eclipse progresses to determine the moments $\mathbf{T_1}$ and $\mathbf{T_2}$ illustrated in Figure 5.

[1] The Greeks figured out the length of the lunar month by careful observations over many years, and by using a long series of records handed down from the Babylonians. Since we don't have that much time (and probably don't know any Babylonians) we are just going to take this number from Hipparcos.

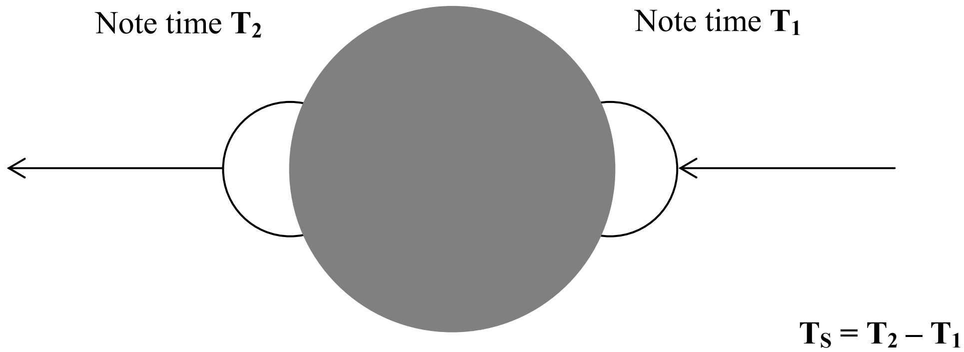


Fig. 5. Timing the shadow crossing.

If we knew the size of the Earth, then we could get the Moon's distance in absolute units (miles or kilometers). Aristarcos didn't have a good number for the size of the Earth, but Hipparcos did, thanks to the geographer Eratosthenes (c. 230 BCE). Eratosthenes determined the difference in latitude, and the difference in distance on the ground, between two cities in Egypt that were on a north-south line. This told him what a degree of latitude amounts to in distance. The Earth's circumference is 360 degrees, or 360 times this distance, and from this he obtained the radius of the Earth in absolute units. Hipparcos then used this figure to get the distance of the Moon.

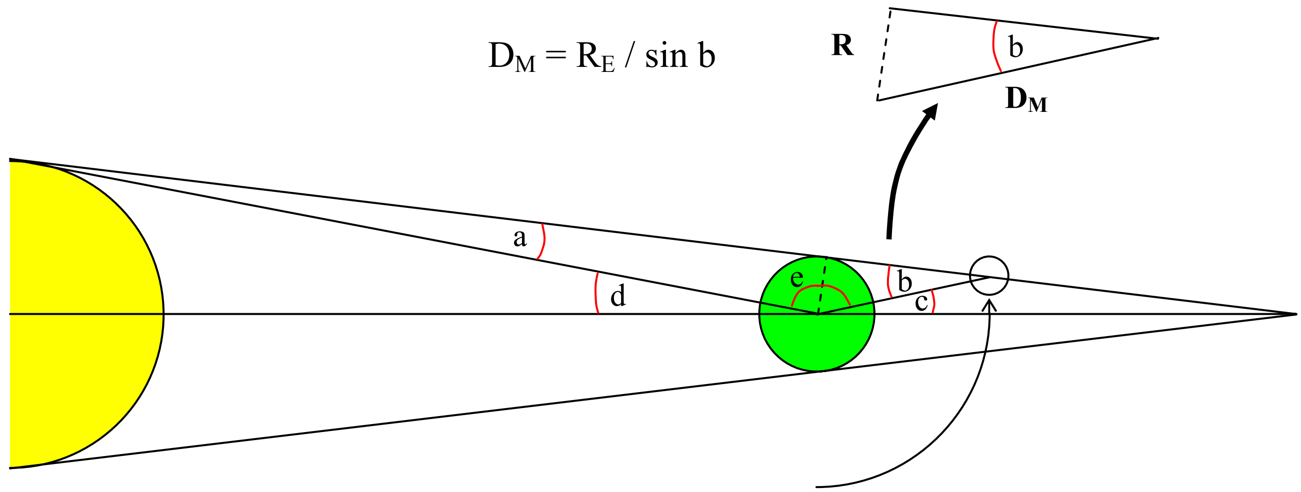
Like Hipparcos, we will use information that other people have determined to finish our calculations. From a map of our state, we'll read off the difference in latitude between two cities on a north-south line, and the distance between them in miles or kilometers. From this we will calculate the radius of the Earth and the distance to the Moon.

The observation log and worksheet for this part of the activity is at the end of this writeup. Keep in mind that we made a simplifying assumption right at the start—that the Earth's shadow is a cylinder, not a cone—so our result will only be approximately correct. Can you identify something else that we are assuming is true when we use this model?

3. Second method: conical shadow model

In this section, we'll improve the accuracy of our distance measurement, and get the size of the Moon too, by starting with a more accurate picture of the geometry—a better model. We'll also build a simple piece of equipment and use it during the daytime to make another necessary measurement. This is often how it goes in science: a better model needs more information to be useful.

We'll use the same geometrical picture that Hipparcos used. Study Figure 6 and its equations carefully and be sure you understand them. The sizes of things and the distances between them are not to scale. This is to make the angles ($\angle$) we will be using easier to see.



$$\angle \mathbf{a} + \angle \mathbf{b} + \angle \mathbf{e} = \mathbf{180\ deg}\ ;\ \angle \mathbf{c} + \angle \mathbf{d} + \angle \mathbf{e} = \mathbf{180\ deg}$$

so

$$\angle \mathbf{a} + \angle \mathbf{b} = \angle \mathbf{c} + \angle \mathbf{d}$$

and

$$\angle \mathbf{b} = \angle \mathbf{c} + \angle \mathbf{d} - \angle \mathbf{a}$$

Fig. 6. The geometry and equations for the conical shadow model

This model is more accurate because it includes the fact that the Earth's shadow is a cone. First look in the upper right corner of Figure 6. The part of the model that includes angle **b** has been enlarged here. Two of the sides of this triangle are the Earth's radius $\mathbf{R_E}$ and the distance to the Moon $\mathbf{D_M}$ **.** If we can determine angle **b** then we can calculate the distance to the Moon, as we did before.

Think about the equations between angles. Angles **a**, **b**, and **e** equal 180 degrees because they form a triangle. Angles **c**, **d**, and **e** equal 180 degrees because they form a straight line. A little algebra shows how to get angle **b** from the other three angles. It isn't obvious just from looking at the figure.

Now think about the angles. Angle **c** is half the angular size of the Earth's shadow where the Moon crosses it –it's the same angle as Θ / 2 in our first model, that we determine by timing the umbral shadow during the eclipse. So we already know how to determine this one. Angle **a** depends on the distance to the Sun. The further away the Sun is, the smaller this angle gets. Notice that angle **a** *doesn't* depend on how big the Sun is, only how far away it is. Finally, angle **d** is half the angular size of the Sun—how big an angle it appears to make in the sky.

We know that the Sun is very far away. We also know that it isn't just a little bright dot in the sky; it has a size that we can easily see, for example by looking at the Sun through thin clouds. So now we are going to make an approximation, just like Hipparcos did: that angle **a** is so much smaller than the other angles that we can neglect it (assume it's zero). Then we have

$$\angle \mathbf{b} \approx \angle \mathbf{c} + \angle \mathbf{d} \qquad (\text{ since } \angle \mathbf{a} \approx 0)$$

So we have to measure the angular size of the Sun and take half of it to get angle **d** . Then we use that and the value for angle **c** from umbral shadow timings to get angle **b** and an improved value for the distance of the Moon.

4. Measuring the Sun with a pinhole camera

To measure the angular size of the Sun we are going to make and use a **pinhole camera**. This is a simple device for forming an image of distant objects without lenses or mirrors. The basic idea depends on the fact that light travels in straight lines. Figure 7 shows how it works.

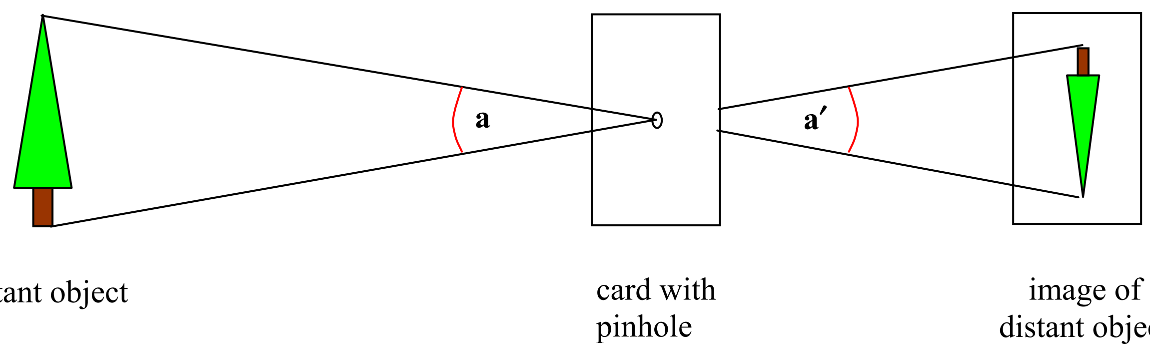


Fig. 7. Principle of the pinhole camera. Angles **a** and **a′** are the same.

The very small hole in an opaque card insures that light from a point on the object can only go to one place on a screen behind it, thus forming an image which we can measure. A key fact for our experiment is that the angular size of the image is the same as the angular size of the object. So we will use a pinhole camera to make an image of the Sun, measure it, and find its angular size.

To construct the camera, we need: two small boxes about the size of shoeboxes, a board or stick four to six feet long, a small piece of aluminum foil, a small piece of white paper, a millimeter ruler, scissors and tape. The boxes will be fastened to the board at either end with their open tops facing each other. A longer board produces a larger image that's easier to measure. Six feet is about the longest board that is easy to handle. But first we need to create the pinhole and the screen.

Figure 8 shows the steps for making the pinhole and the screen. For the pinhole, start by cutting an opening about an inch square in the middle of the bottom of one box. This doesn't have to be very accurate. Next, cut out a square of aluminum foil about two inches on a side. Use something with a sharp, round point to make a very small hole in the middle of the foil square. This tool could be a pin or needle, a very sharp pencil point, or the metal point of a compass (the kind used to make circles, not to find North!). The pinhole should be about one millimeter in diameter, very round, with clean sharp edges. You may have to make several holes on different pieces of foil in order to get a really good one. When you have a good pinhole, measure its diameter with the millimeter ruler and enter this number on the worksheet. A magnifying glass helps with this, if you have one. Finally, tape the foil over the opening cut into the box bottom, centering the pinhole in the opening by eye.

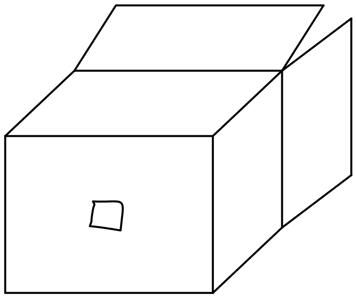
Making the screen is easier. Cut out a piece of white paper about two inches square. Draw a line across the middle of it. Using the ruler, make a mark on the line every 5 millimeters. Tape the paper into the inside bottom of the other box, in the middle.

Now fasten the two boxes to the board or stick so the screen faces the pinhole. The sides of the box around the screen help to shade the image from ambient daylight and make it easier to see. Use the ruler to measure the distance between the pinhole and the screen along the board. Enter this number on the worksheet. Figure 9 illustrates this step.

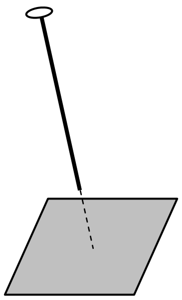
Figure 10 shows how to use the camera. To make the measurement, we need to get the pinhole image of the Sun onto the screen. On a clear day, take the camera outdoors and point the pinhole end at the Sun. Move it around so the shadow of the upper (pinhole) box falls on the lower (screen) box. You should see the solar image somewhere inside the lower box. Move the camera around to center this image on the screen, and measure its diameter with the homemade ruler, estimating between the marks to the nearest millimeter (or finer than that, if you think you can). It helps to prop the camera board against something—a chair back, a fence, a garbage can—and rest the screen end on the ground, then move the bottom end around carefully until the image is centered on the scale. Enter the diameter of the solar image on the worksheet. You'll see the image moving along the scale even with the camera setting still. Do you know why this happens?

Now there's a small correction to make: subtract the diameter of the pinhole from the diameter of the image. This gives the diameter of the image that we would measure if the pinhole were infinitely small. Do you see why this is necessary?

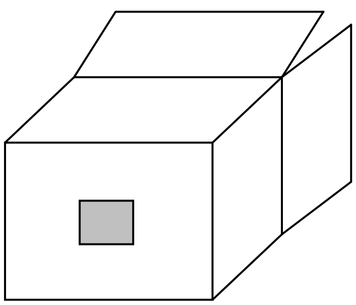
We now have two length measurements, the (corrected) diameter of the solar image, and the distance from the pinhole to the screen. These form an angle, the angular size of the Sun, which we can calculate. This gives us everything we need to finish the improved calculation of the distance of the Moon, following the worksheet. How does the new result compare with the original result?



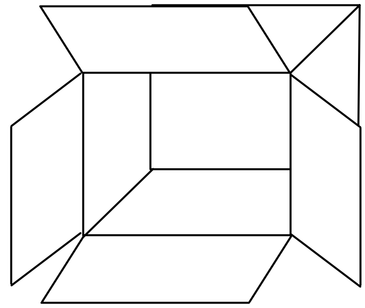
1. Cut an opening in a small box.



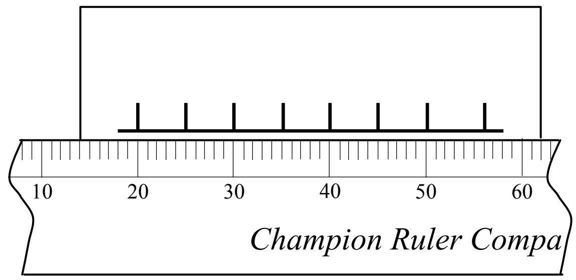
2. Make a pinhole in a piece of foil.



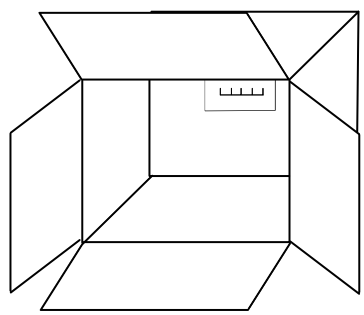
3. Tape the pinhole over the opening.



4. Open up a similar sized box.



5. Copy a millimeter scale onto a piece of paper.



6. Tape the paper scale into the bottom of the second box.

Fig. 8. Steps in making the pinhole and the ruled screen for the pinhole camera.

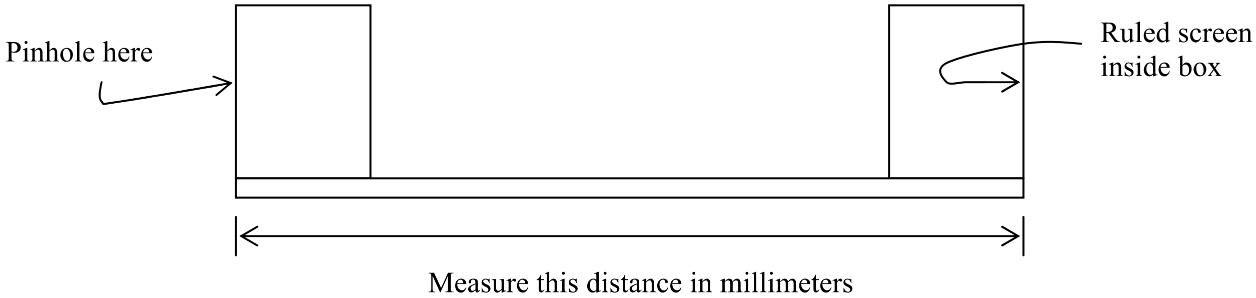


Fig. 9. Assembling the pinhole camera.

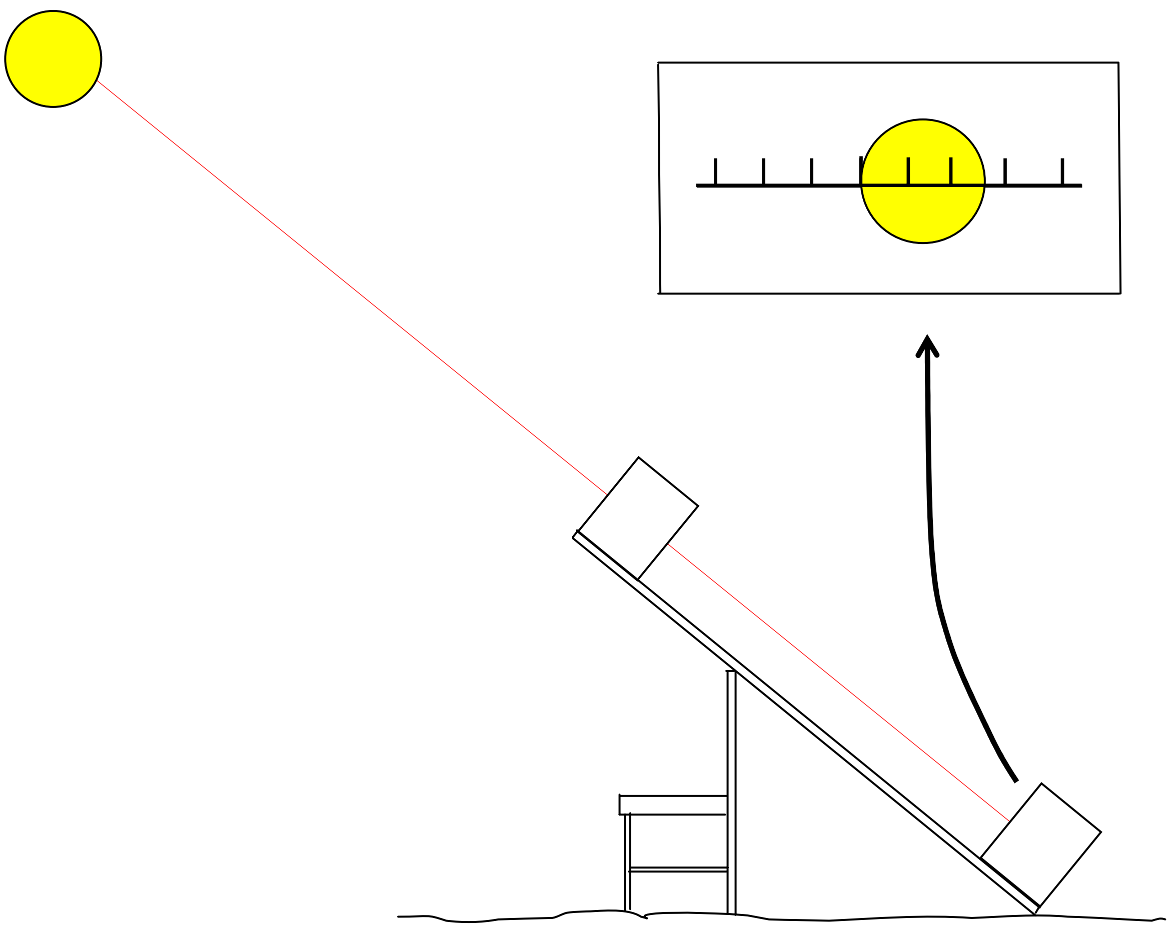
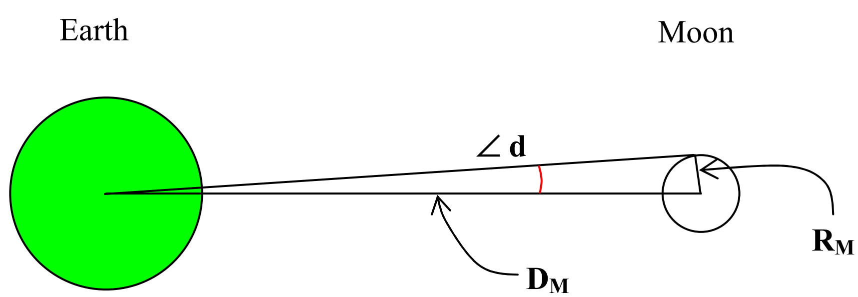


Fig. 10. Using the pinhole camera. The inset, upper right, shows how to use the scale to measure the size of the solar image. You can estimate this to the nearest millimeter.

4. Determining the size of the Moon

As a final calculation, we can determine the size (the radius) of the Moon using what we've measured and calculated. This is because of an intriguing fact of Nature: the angular size of the Moon is the same as the angular size of the Sun. There's no particular reason why this should be true, but we know it is by observing eclipses of the Sun. During a **solar eclipse** the Moon barely covers the Sun. If you're not at exactly the right place on Earth to be in the shadow, you will see a little sliver of the Sun peeking out somewhere around the edge.

The geometry and equation for calculating the radius of the Moon are shown in Figure 11. The angle **d** is the same as in Figure 6 for the Sun. The distance $\mathbf{D_M}$ we've already calculated. So the radius of the Moon $\mathbf{R_M}$ is easy to find. You calculated the radius of the Earth $\mathbf{R_E}$ already. How big is the Moon compared to the Earth?



$$R_M / D_M = \sin d \quad \text{so} \quad R_M = D_m \times \sin d$$

Fig. 11. Calculating the radius of the Moon.

Worksheet for the Cylindrical Shadow Method

Location and date of eclipse observations: ____

Time $\mathbf{T_1}$ at which the umbra crossed the center of the Moon: ____________

Time $\mathbf{T_2}$ at which the umbra crossed the center of the Moon: ____________

Time difference $\mathbf{T_s = T_2 - T_1}$, in minutes: ____________

Lunar month $\mathbf{L_M}$ in minutes, 27.322 days x 24 hours/day x 60 minutes/hour: ____________

Angular size of the Earth's shadow $\mathbf{\Theta = 360\ deg \times [\,T_s\ /\ L_M\,]}$: ____________

$\Theta / 2$ (degrees): ________ Sin [$\Theta / 2$] : ________ 1 / sin [$\Theta / 2$] : ____________

Distance of the Moon $\mathbf{D_M}$ in units of Earth radius,
$\mathbf{D_M = R_E \times \{ 1 / \sin [\, \Theta / 2 \,] \}}$: ____________

Latitude $\mathbf{L_N}$ of northern city in degrees, read from map: ____________

Latitude $\mathbf{L_S}$ of southern city in degrees, read from map: ____________

Latitude difference in degrees, $\mathbf{L_N - L_S}$:

____

Distance **d** between the two cities in miles or kilometers, read from map: ____________

Distance per degree of latitude, $\mathbf{d / [L_N - L_S]}$: ____________

Circumference of the Earth, $\mathbf{C_E = 360\ deg \times \{ d / [L_N - L_S] \}}$: ____________

Radius of the Earth, $\mathbf{R_E = C_E / 2\pi}$: ____________

Distance of the Moon in miles or kilometers, $\mathbf{D_M \times R_E}$: ____________

Can you think of another assumption or approximation that we've made in this experiment?

Worksheet for the conical shadow method

Angle **c** = angle **Θ / 2** from the first worksheet: ____________

Radius of the Earth $\mathbf{R_E}$, in miles or kilometers, from the first worksheet: ____________

Location, date, and time of pinhole camera observations: ____________________________________

Diameter of pinhole **p** in millimeters: ____________

Distance from pinhole to screen **L** in millimeters: ____________

Diameter of solar image **s** in millimeters: ____________

Corrected diameter of solar image **S** = **s – p** in millimeters: ____________

Angular diameter **D** of Sun: sin **D** = **S / L** : _______________ **D** (degrees) : ____________

Why does the solar image move across the screen? ______________________________________

Angle **d** = **D / 2** : ____________

Angle **b** = angle **c** + angle **d** : ____________

Sin **b** : __________________ 1 / sin **b** : ____________

Distance of the Moon $\mathbf{D_M}$ = $\mathbf{R_E}$ **x { 1 / sin b } :** ____________

Compare this value for $\mathbf{D_M}$ with your first result: ____

Angle **d** : ____________ sin **d** : ____________

Radius of the Moon, $\mathbf{R_M}$ = $\mathbf{D_M}$ **x sin d :** ____________

Compare the size of the Moon and the size of the Earth: ___________________________________